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Dimensional Reduction for Filters of Nonlinear Systems with Time-Scale Separation

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This project outlines a collection of problems which combine techniques of model reduction and filtering. The basis of this work is a collection of limit theories for stochastic processes which model dynamical systems with multiple time scales. These different time scales often allow one to find effective behaviors of the fast time scales. When the rates of change of different variables differ by orders of magnitude, efficient data assimilation can be accomplished by constructing nonlinear filtering equations for the coarse-grained signal. In particular, we study how scaling interacts with filtering via stochastic averaging. We combine our study of stochastic dimensional reduction and nonlinear filtering to provide a rigorous framework for identifying and simulating filters which are specifically adapted to the complexities of the underlying multi-scale dynamical system.

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DIMENSIONAL REDUCTION FOR FILTERS WITH TIME-SCALE SEPARATION

AFOSR GRANT NUMBER FA9550-08-1-0206

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Abstract
This project outlines a collection of problems which combine techniques of model reduction and filtering. The basis of this work is a collection of limit theories for stochastic processes which model dynamical systems with multiple time scales. Multiple time scales occur in many real systems, and reflect different orders of magnitudes of rates of change of different variables. These different time scales often allow one to find effective behaviors of the fast time scales. In systems subject to both bifurcations and noise (which form one of the main components of this project), various *singular perturbations* problems must be understood. When the rates of change of different variables differ by orders of magnitude, efficient data assimilation can be accomplished by constructing nonlinear filtering equations for the coarse-grained signal. We consider the conditional law of a signal given the observations in a multi-scale context. In particular, we study how scaling interacts with filtering via stochastic averaging. We combine our study of stochastic dimensional reduction and nonlinear filtering to provide a rigorous framework for identifying and simulating filters which are specifically adapted to the complexities of the underlying multi-scale dynamical system.

This is the final report for AFOSR GRANT NUMBER FA9550-08-1-0206, which was awarded in August 2008 and ended December 2011. This research involved the work of the PI, N. Sri Namachchivaya, Co-PI, Richard B. Sowers, and graduate students at University of Illinois at Urbana-Champaign (UIUC) in collaboration with a graduate student at the Humboldt University, Berlin, Germany.

Introduction

The first objective of current project is concerned with certain methods of *dimensional reduction* of nonlinear systems with symmetries and small noise [1,2]. In the presence of a separation of scales, where the noise is asymptotically small, one exploits symmetries to use recent mathematical results concerning *stochastic averaging* to find an appropriate lower-dimensional description of the system. Reduced models can be used in place of the original complex models, either for simulation and prediction or real-time control. To this end, reduced models [3,4]. often provide qualitatively accurate and computationally feasible descriptions.

The second objective is to derive a low-dimensional filtering equation, that determines conditional law of a plant, in a multi-scale environment given the observations. This project is less concerned with specific applications and more focused on some of the theoretical aspects that deal with reduced dimensional nonlinear filters. In particular, we showed the efficient utilisation of the low-dimensional models of the signal to develop a low-dimensional filtering equation.

We combine two ingredients, namely, stochastic dimensional reduction discussed above and nonlinear filtering [6,7]. We achieved this through the framework of homogenisation theory which enables us to average out the effects of the fast variables.

To introduce the basic idea of filtering in a multi-scale environment, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. We consider nonlinear $\mathbb{R}^m \times \mathbb{R}^n$-valued signal processes $(Z^\varepsilon, X^\varepsilon)$ and an $\mathbb{R}^d$-valued observation process Y^ε given by the SDE's

$$\begin{aligned} dZ_t^\varepsilon &= \varepsilon^{-1} a(Z_t^\varepsilon, X_t^\varepsilon)dt + \varepsilon^{-1/2}\gamma(Z_t^\varepsilon, X_t^\varepsilon)dW_t, \quad Z_0^\varepsilon = \eta \\ dX_t^\varepsilon &= b(Z_t^\varepsilon, X_t^\varepsilon)dt + \sigma(Z_t^\varepsilon, X_t^\varepsilon)dV_t, \quad X_0^\varepsilon = \xi \\ dY_t^\varepsilon &= h(Z_t^\varepsilon, X_t^\varepsilon)dt + dB_t, \quad Y_0^\varepsilon = 0 \end{aligned} \tag{1}$$

where W, V and B are independent Wiener processes and η and ξ are random initial conditions which are independent of W, V and B. Let ε be a small parameter that measures the ratio of slow and fast time scales. Hence the dynamics of (1) are separated into two scales, where Z^ε and X^ε represent the fast and slow variables, respectively. The *generator* $\mathscr{L}_\varepsilon$ of the Markov processes $(Z^\varepsilon, X^\varepsilon)$ is then of the form

$$\mathscr{L}_\varepsilon \varphi = \varepsilon^{-1}\mathscr{L}_F \varphi + \mathscr{L}_S \varphi, \tag{2}$$

for all $\varepsilon \in (0,1)$ and all $\varphi \in C^\infty(\mathbb{R}^{m+n})$, where $\mathscr{L}_F$ and $\mathscr{L}_S$ represent generators of fast and slow variables.

The main objective of filtering theory is to estimate the statistics of the signal $\mathfrak{X}_t^\varepsilon \overset{\text{def}}{=} (Z_t^\varepsilon, X_t^\varepsilon)$ at time t based on the information in the observation process Y^ε up to time t, that is, on the basis of the sigma-algebra, $\mathscr{Y}_t^\varepsilon$. More precisely for each $t \geq 0$, we want to find the conditional *law* of $\mathfrak{X}_t^\varepsilon$ given $\mathscr{Y}_t^\varepsilon$, that is, we want to compute $\mathbb{P}\{\mathfrak{X}_t^\varepsilon \in A | \mathscr{Y}_t^\varepsilon\}$ for all $A \in \mathscr{B}(\mathbb{R}^{m+n})$. The insight of filtering theory is that one can construct this conditional measure via a stochastic PDE which is a recursive equation driven by the observation process.

If the above conditional measure admits a smooth density, $p^{\varepsilon,\mathfrak{X}}(t,\mathfrak{x})$ with respect to Lebesgue measure i.e., $\mathbb{P}\{\mathfrak{X}_t^\varepsilon \in A|\mathscr{Y}_t^\varepsilon\} = \int_A p^{\varepsilon,\mathfrak{X}}(t,\mathfrak{x})d\mathfrak{x}$ for all $A \in \mathscr{B}(\mathbb{R}^{m+n})$, then $u^\varepsilon(t,\mathfrak{x})$, the *un-normalised density*, solves the Zakai equation

$$du^\varepsilon(t,\mathfrak{x}) = \mathscr{L}_\varepsilon^* u^\varepsilon(t,\mathfrak{x})dt + u^\varepsilon(t,\mathfrak{x})h(\mathfrak{x})dY_t^\varepsilon, \quad u^\varepsilon(0,\cdot) \ = p_{\mathfrak{x}}, \tag{3}$$

where $\mathscr{L}_\varepsilon^*$ is the adjoint operator of the $\mathscr{L}_\varepsilon$ (with respect to Lebesgue measure on $\mathbb{R}^{m+n}$) and $\mathfrak{X}_0^\varepsilon$ has density $p_{\mathfrak{x}}$.

Since we are interested in the slowly-varying coordinates which usually describe the essential coarse-grained dynamics, the focus of our work is to estimate the signal X_t^ε at time t on the basis of the sigma-algebra $\mathscr{Y}_t^\varepsilon$. More precisely for each $t \geq 0$, we want to find the conditional *law* $\mathbb{P}\{X_t^\varepsilon \in A|\mathscr{Y}_t^\varepsilon\}$ for all $A \in \mathscr{B}(\mathbb{R}^n)$. For each $t \geq 0$, define the $C([0,t];\mathbb{R}^d)$-valued random variable $\vec{\mathbf{Y}}_{[0,t]}^\varepsilon$ as $(\vec{\mathbf{Y}}_{[0,t]}^\varepsilon)_s(\omega) \overset{\text{def}}{=}$

$\mathbf{Y}^\varepsilon_s(\omega)$ for all $\omega \in \Omega$ and $s \in [0,t]$. The point of filtering is that for each $t \geq 0$ and $\varepsilon \in [0,1)$, there is a measurable map

$$\pi^\varepsilon_t : C([0,t];\mathbb{R}^d) \to \mathscr{P}(\mathbb{R}^n) \tag{4}$$

such that for each $t \geq 0$ and $A \in \mathscr{B}(\mathbb{R}^n)$, $(\pi^\varepsilon_t(\vec{\mathbf{Y}}^\varepsilon_{[0,t]}))(A) = \mathbb{P}\{X^\varepsilon_t \in A | \mathscr{Y}^\varepsilon_t\}$.

Informally, our focus is in the behavior of the filters when system (1) can be averaged with respect to $\mu_x(dz)$, the unique invariant measure of the fast variable Z^ε_t, i.e., $\mathscr{L}^*_F \mu_x = 0$. Notice that the dimension of the signal (the coarse-grained dynamics) is reduced to n. Letting $u^0(t,x)$ satisfy the Zakai equation for the reduced system

$$du^0(t,x) = \bar{\mathscr{L}}^*_S u^0(t,x)dt + u^0(t,x)\bar{h}(x)dY^0_t,$$

one might hope that u^ε converges to u^0. (u^0 does not depend on fast variable z.)

As in (4), there is a map

$$\pi^0_t : C([0,t];\mathbb{R}^d) \to \mathscr{P}(\mathbb{R}^n)$$

such that $(\pi^0_t(\vec{\mathbf{Y}}^0_{[0,t]}))(A) = \mathbb{P}\{X^0_t \in A | \mathscr{Y}^0_t\}$. However, we want to use the original observation $\mathbf{Y}^\varepsilon$ instead of $\mathbf{Y}^0$ to get a filter for the reduced system since $\mathbf{Y}^\varepsilon$ is the actual data we would obtain. Our main claim [9] is

Theorem 0.1. *For each $t > 0$,*

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[d_{\mathscr{P}(\mathbb{R})}(\pi^\varepsilon_t(\cdot, \vec{\mathbf{Y}}^\varepsilon_{[0,t]}), \pi^0_t(\cdot, \vec{\mathbf{Y}}^\varepsilon_{[0,t]}))\right] = 0.$$

Here $d_{\mathscr{P}(\mathbb{R})}$ is the standard Prohorov metric on $\mathscr{P}(\mathbb{R})$, and the statement is convergence in probability. In other words, if you are only interested in finding the conditional distribution of the slow variables, you might as well use the averaged dynamics.

Homogenized Hybrid Particle Filter (HHPF):

Particle methods have a long history; we refer the reviewer to [7] for an extensive treatment of this area. The basic idea is to replace the expected values of the quantities in the statement of nonlinear filters, by suitable Monte-Carlo sample averages. The basic requirements for a particle method are the simulation of independent samples of the signal, called particles, with the same stochastic law as the signal, and the re-sampling of these particles to incorporate the information from the observations.

Our approach described here, not only reduces the computational burden for real time applications but also helps solve the problem of particle degeneracy. In the case where the signal solves a stochastic differential equation, the simulation of each particle's SDE is usually implemented with discrete time Euler or Miltsein approximations. The conditional distribution of the signal at the current time t is approximated by the locations of the particles and their weights, specifically. The set of particles are then adjusted in some manner to conform to each observation by either assigning each particle a scalar weight value, $w_j^n(t)$ that is calculated for the j^{th} particle at $(x_t^{\varepsilon,j}, z_t^{\varepsilon,j})$ to account for the information from the observation up to time t by interchanging the state values associated with each particle in some manner or by cautious novel branching techniques presented here.

An Illustrated Example:

In this section, we provide some evidence that suggests that the proposed approaches are likely to work, with sufficient accuracy, in practice. We apply the HHPF to an example to illustrate its potential for high-dimensional complex problems.

Consider the following signal model:

$$\dot{Z}_t^\varepsilon = -\frac{1}{\varepsilon}(Z_t^\varepsilon - X_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}}\dot{W}_t, \quad Z_0^\varepsilon = z_0, \qquad \dot{X}_t^\varepsilon = -(Z_t^\varepsilon)^3 + \sin(\pi t) + \cos(\sqrt{2}\pi t), \quad X_0^\varepsilon = x_0$$

with the observation

$$Y_t^\varepsilon = \frac{1}{2}(X_s^\varepsilon)^2 + B_t.$$

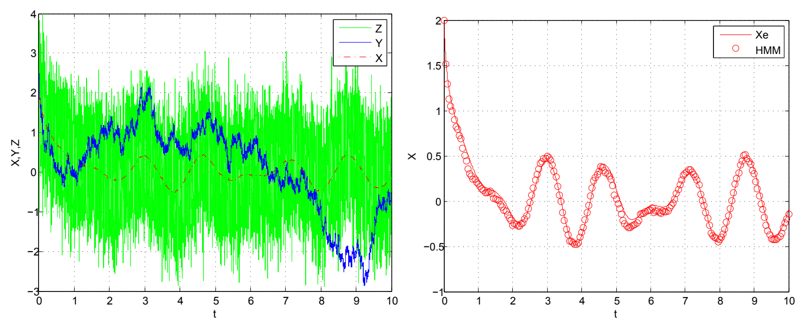

FIGURE 1. Signal and Observation (left) and HMM solution and the original Signal (right)

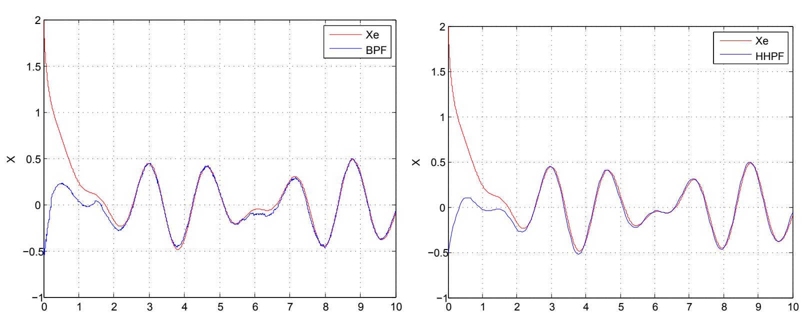

FIGURE 2. Full branching particle filter (left) and HHPF (right)

Figure 1 (left) shows a typical plot for this multiscale system with slow and fast signals, X_t^ε and Z_t^ε, respectively, and with the observations Y_t^ε. We then applied the algorithms described above to the this problem using MATLAB. Figure 1 (right) compares the HMM solution with the analytical solution. The results from the branching particle filter and the HHPF are given in Figures 2 (left) and 2 (right), respectively. The time taken for these simulations are 448 sec. and 15 sec. respectively with a 2 GHZ Intel Core 2 Duo MacBook. The parameters we have used are: $N = 200$, $M = 2$, $n_T = 10$, $N_m = 200$, $\delta t = 1e - 4$, $\Delta t = 5e - 2$, $\varepsilon = 1e-3$. Here n_T is the number of micro time steps skipped to ignore transients and N_m is the number of micro time steps. These preliminary results are critical in order to demonstrate that the proposed methods show promise.

The theoretical aspect of data assimilation was accomplished by constructing nonlinear filter equations for the coarse-grained signal (i.e., the lower dimensional Zakai equation), as explained in [8, 9]. We showed that a multiscale particle method could be effectively applied to large dimensional systems. The separation of time-scales facilitated the construction of the particle filter which allows the dimensional reduction of the system needed to solve for each particle.

In summary, this novel computational method combined homogenization of random dynamical systems, the results of [8, 9] on reduced order nonlinear filtering, with sequential Monte Carlo methods to reduce the effective number of variables in the evaluation of the conditional distribution needed in the Bayesian filter for data assimilation. For computational purposes, the conditional distribution $\bar{\pi}$ is approximated by particle filters; the system of particles is resampled in the extreme cases described in the previous paragraph. It is worthwhile to note that the convergence results of both stochastic averaging and the branching particle filter are obtained through the method of the martingale problem. Thus, it is natural to consider the convergence result of HHPF in the framework of the martingale problem, which is left for future work.

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